markedly with injection rate. For $I=0.01$ the recirculation bubble extends from about 1 to 4 base radii downstream of the base. Radial surveys slightly upstream of this recirculation bubble reveal a uniform bleed flow extending over 90% of the base diameter. For $I=0.037$ the recirculation bubble is blown off, and the axial pressure gradients along the wake are extremely small. Beyond $I=0.037$ only a small increase in the near-wake static pressure can be expected, since it must approach P_l as I approaches 1.

With external compression, the near-wake length is predominately controlled by the length scale of the compression region. Again, for $I=0.037$ the axial pressure gradients are small and apparently the wake is nearly fully opened by the bleed flow. Although a weak recirculation bubble is indicated by the data, this is not within the demonstrated $\pm 1\%$ data accuracy. The rapid acceleration the centerline velocity to sonic speed is aided by the externally generated expansion simulating a tailoff of combustion as well as increased turbulent mixing.

Acknowledgment

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A Modified One-Equation Model of Turbulence for the Calculation of Free Shear Flows

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Nomenclature

C_D = coefficient of dissipation of turbulent kinetic energy
k = turbulent kinetic energy
k_m = maximum value of k at certain x
h_s = empirical constant, $h_s = \Delta\delta$
ℓ = length-scale of turbulence
u = local mean velocity in x direction
U = freestream velocity
x,y = distances measured in the mainstream direction and the direction normal to it respectively
x_i,x_f = initial and final values of x for flow region considered
ρ = density
$\nu_{\rm eff}$ = effective viscosity
δ = boundary-layer thickness defined at $u = U + 0.01\epsilon$
$\delta_{1/2}$ = value of y where $u = U + 0.5\epsilon$
τ = local shear stress (turbulent)
λ = nondimensional length-scale of turbulence, $= \ell/\delta$
σ_k = Schmidt number of turbulent kinetic energy
Δ = boundary-layer thickness for the linear approximation of the velocity profile
ϵ = maximum velocity difference across the layer

Subscripts
i = condition at x_i

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Introduction

THE use of a turbulence model to evaluate the effective viscosity is common in the calculation of free shear flows. Models generally describe this quantity as proportional to the product of the length-scale and a velocity-scale of turbulence

$$\nu_{\rm eff} = \ell k^{1/2} \tag{1}$$

The molecular viscosity is neglected in Eq. (1), and the length-scale is defined such that the proportionality constant is unity. Values of ℓ and k are to be determined either empirically or by solving certain conservation equations. Models that require the solution of zero, one, or two equations are well known and have been used for the calculation of different turbulent flows.[1-5] As the number of equations is increased, the generality of a turbulence model is generally improved on at the expense of its simplicity.

The one-equation model employs an empirical variation for the length-scale of turbulence. The present work is to improve the generality of this model by the derivation of a semiempirical expression for the length-scale using an integral analysis, with some approximations, applied to two-dimensional free shear flows.

Analysis

The length-scale of turbulence can be expressed as:

$$\ell = \lambda\delta \tag{2}$$

where λ is normally taken as an empirical constant. This expression can be improved if λ is related to the mean flow characteristics; to obtain such relation, the integral forms of the governing equations are analytically solved. In order to simplify the algebraic solution, approximate shapes are assumed for different profiles to evaluate the integral quantities. A linear mean velocity profile is assumed, while the turbulent kinetic energy profile is taken uniform across the layer. If we consider the general shape of the velocity profile for a flow remote from walls, it could be linearly approximated as shown in Fig. 1. We may consider the relation

$$\Delta = h_s\delta \tag{3}$$

where h_s is a quantity of the order of unity whose value is to be determined later.

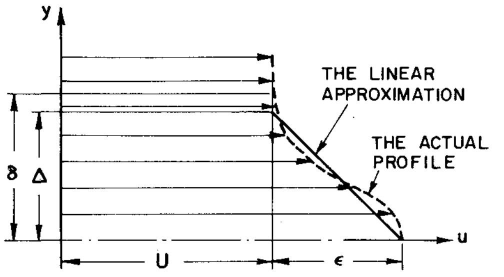

Fig. 1 Linear approximation of the velocity profile assumed for the derivation of an expression for λ.

The present analysis includes both plane and axisymmetric flows. For plane flows, the following equations represent, respectively, the y-wise integrals of the parabolic equations for momentum, mean motion kinetic energy, and turbulence energy:

$$\tfrac{1}{2}\epsilon\Delta\frac{dU}{dx}+\frac{d}{dx}\left[\left(\frac{\epsilon U}{2}+\frac{\epsilon^2}{3}\right)\Delta\right]=0 \tag{4}$$

$$\frac{d}{dx}\left[\left(U^2+\epsilon U+\frac{\epsilon^2}{4}\right)\frac{\epsilon\Delta}{2}\right]=-\frac{\tau}{\rho}\epsilon \tag{5}$$

$$\frac{d}{dx}\left[k\Delta\left(U+\frac{\epsilon}{2}\right)\right]=k^{1/2}\ell\frac{\epsilon^2}{\Delta}-C_D\frac{k^{3/2}}{\ell}\Delta \tag{6}$$

Using Eq. (1), the shear stress can be expressed as:

$$\tau/\rho=k^{1/2}\ell(\epsilon/\Delta) \tag{7}$$

If we assume a balance between generation and dissipation of turbulent kinetic energy, the left-hand side of Eq. (6) can be neglected compared to its right-hand side. This assumption is shown in Ref. 5 to be equivalent to neglecting the ratio (k/ϵ^2), which is usually very small, particularly for flows where ϵ is of the same order of magnitude as U. In the present analysis, the left-hand side of Eq. (6) is not completely neglected, rather, it is approximately evaluated for flows with constant freestream velocity, with the assumption that the streamwise variations of the two following quantities are negligible compared to their arguments

$$\eta=k/\epsilon^2 \tag{8}$$

$$\xi=\epsilon/U \quad \text{for } U\neq 0 \tag{9}$$

If τ is eliminated between Eqs. (5) and (7), the resulting equation and Eq. (4) could be simplified and combined to give the following expression for the left-hand side of Eq. (6):

$$\frac{d}{dx}\left[k\Delta\left(U+\frac{\epsilon}{2}\right)\right]=\frac{24\eta+12\eta\xi}{4+3\xi}\left(-k^{1/2}\ell\frac{\epsilon^2}{\Delta}\right) \tag{10}$$

Substituting Eq. (10) into Eq. (6) we obtain an expression for k that could be used with Eqs. (2-5) and (7) to yield the following expression for λ:

$$\lambda^2=h_s^3\left[\frac{C_D(4+3\xi)}{4+3\xi+24\eta+12\eta\xi}\right]^{1/2}\left[-\left(\frac{2U+\epsilon}{12\epsilon U+8\epsilon^2}\right)\delta\frac{dU}{dx}-\left(\frac{12U^2+18\epsilon U+6\epsilon^2}{72\epsilon^2U+48\epsilon^3}\right)\delta\frac{d\epsilon}{dx}\right] \quad \text{for } U\neq 0 \tag{11}$$

For flow cases with zero freestream velocity, ξ is not defined, and the following expression for λ can be similarly obtained

$$\lambda^2=-h_s^3\left(\frac{C_D}{1+4\eta}\right)^{1/2}\frac{1}{8\epsilon}\delta\frac{d\epsilon}{dx}, \quad \text{for } U=0 \tag{12}$$

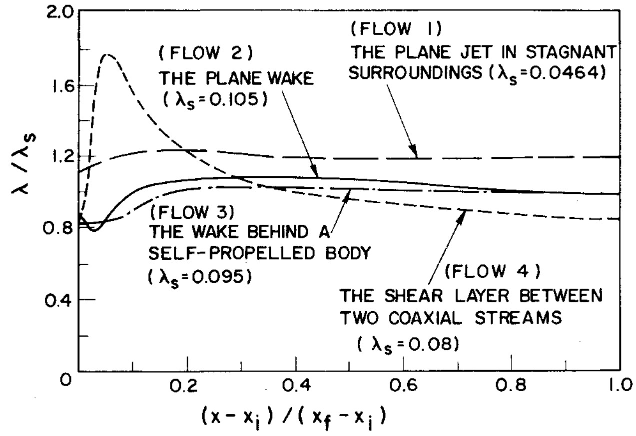

Fig. 2 Calculated values of λ in the downstream direction as compared to values selected previously for best results, λ_s.

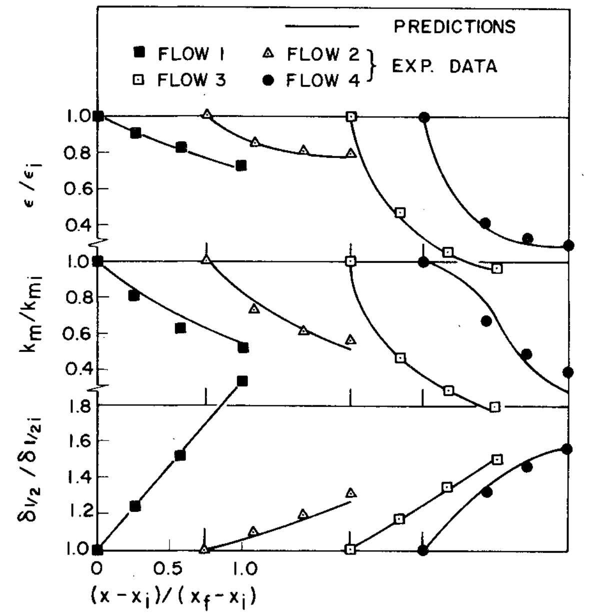

Fig. 3 Predictions of characteristics of free shear flows.

For axisymmetric flows, the proper forms of the governing equation are used to obtain the following relations:

$$\lambda^2=h_s^3\left[\frac{C_D(5+3\xi)}{5+3\xi+120\eta+20\eta\xi}\right]^{1/2}\left[-\left(\frac{10U+3\epsilon}{120\epsilon U+60\epsilon^2}\right)\delta\frac{dU}{dx}-\left(\frac{10U^3+12\epsilon U+3\epsilon^2}{120\epsilon^2U+60\epsilon^3}\right)\delta\frac{d\epsilon}{dx}\right] \quad \text{for } U\neq 0 \tag{13}$$

$$\lambda^2=-h_s^3\left(\frac{C_D}{1+6.67\eta}\right)^{1/2}\frac{\delta}{20\epsilon}\frac{d\epsilon}{dx} \quad \text{for } U=0 \tag{14}$$

Results

The expression obtained for λ is incorporated with the one-equation model to obtain predictions for the following flows where experimental data are available in the literature: flow 1—plane jet in stagnant surroundings, experimental data by Heskestad[6]; flow 2—plane wake in a steady stream, experimental data by Townsend[7]; flow 3—round wake behind a self-propelled body in a steady stream, experimental data by Naudascher[8]; flow 4—shear layer between two round coaxial streams, experimental data by Zawacki et al.[9]

The finite-difference method of Patankar and Spalding[4] was used for the solution of equations of continuity, momentum, and turbulence energy. The set of equations includes three constants whose values were taken as: $h_s = 0.932$ (which gives $h_s^{3/2} = 0.9$), $\sigma_k = 2.0$, and $C_D = 0.3$ for flows where $\epsilon \ll U$, and 0.08 for flows with ϵ of the same order as U. Different values reported in the literature for the last two constants are quoted by Launder and Spalding,[1] ranging from 0.06 to 0.1 for C_D and from 1.0 to 2.0 for σ_k.

Predictions were obtained initially with the value of λ selected for each flow to offer the best possible agreement with experiment. The selected values are 0.0464 for flow 1, 0.105 for flow 2, 0.095 for flow 3, and 0.08 for flow 4. These values are labeled λ_s and are in fair agreement with the literature, where a value of 0.04 was used by Emmons[2] for the plane jet, and of 0.06 was used by Sechagiri[3] for the plane interacting wakes. Figures 2 and 3 present prediction results obtained using the λ equation. Figure 2 shows the streamwise variations of the calculated λ for each flow in relation to λ_s. It can be noticed that the value of (λ/λ_s) is close to unity for all flows if we discard values obtained for the first few steps of the marching integration calculation, where results are much influenced by the accuracy of initial conditions. Although values of λ_s vary from 0.0464 to 0.105 for flows considered, the maximum deviation of (λ/λ_s) from unity is about 20%.

Figure 3 shows predictions of the streamwise variations of the following characteristics for each flow: maximum velocity difference across the flow ϵ, maximum value of turbulent kinetic energy k_m, and boundary-layer thickness $\delta_{1/2}$. In general, agreement between predictions and experiment is satisfactory for all flows considered. This agreement is generally better for mean velocity values as compared to values of turbulent kinetic energy, which is well expected regarding the larger inaccuracies associated with the experimental measurements of the later quantity and the amount of empirical input in its conservation equation.

Conclusions

The semiempirical relation obtained for the length-scale of turbulence has potentially improved the universality of the one-equation model. With the same values of empirical constants, the model provided acceptable predictions of both mean and turbulent quantities for four flows with different geometries. The model might therefore be recommended for the prediction of two-dimensional free shear flows. The advantage of the two-equation model over the present model is not expected to be significant enough to justify solving one extra partial differential equation and dealing with five or six empirical constants instead of three.

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Derivation of an Integral Equation for Three-Dimensional Transonic Flows

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Introduction

HEASLET and Spreiter[1] derived the integral equation for steady, inviscid, three-dimensional transonic flows. They stated that the field point was surrounded by a sphere of infinitesimal radius, but in the subsequent analysis they actually surrounded the field point by an infinite strip of infinitestimal thickness in the streamwise direction. In this Note, we surround the field point by a hexahedron and derive an integral equation from which the equations by Heaslet and Spreiter[1] can be deduced as a special case. The method is similar to that used by Ogana and Spreiter[2] to derive the integral equation for two-dimensional transonic flows. Although numerical methods are not analyzed here, the approach used by Ogana[3] can be applied to obtain numerical solutions to the integral equation derived by the present method.

Mathematical Preliminary

The mathematical concepts essential to the analysis are briefly presented in this section.

1) Consider the question of convergence of a volume integral. Let f be a function which becomes infinite at a point P within the volume V, then f is singular at P. To define the integral of f throughout the volume V, we surround the singular point P by a small enclosed cavity σ, then let σ vanish while always surrounding P, that is[4]:

$$\int_V f \mathrm{d}V \equiv \lim_{\sigma \to 0} \int_{V-\sigma} f \mathrm{d}V \tag{1}$$

The volume integral is said to be convergent if the limit on the right-hand side of Eq. (1) is finite; otherwise, it is divergent. If the integral is convergent, but the value of the limit depends on the shape of σ, it is said to be semiconvergent.[4]

Convergence of the volume integral is guaranteed if within a sphere of finite radius, whose center is P, the function f satisfies the inequality:

$$|f| < Mr^{-\mu} \tag{2}$$

where $\mu < 3$, M is a definite constant, and r is the distance between P and the point at which f is estimated. For divergence, $\mu \geq 3$, but semiconvergence may occur if $\mu = 3$.

2) Now, consider Green's theorem when some of the integrals may be semiconvergent and also consider the differentiation of a volume integral with respect to a parameter. Let the function Ω, together with its first- and second-order derivatives, be finite throughout the volume V. In addition, let ψ be a function which becomes infinite at a point P in the volume V. Green's theorem,[4] applied to the region bounded internally by σ and externally by a surface S, is:

$$\int_V (\psi \nabla^2 \Omega - \Omega \nabla^2 \psi) \mathrm{d}V = -\int_S \left(\psi \frac{\partial \Omega}{\partial n} - \Omega \frac{\partial \psi}{\partial n}\right) \mathrm{d}S$$

$$-\lim_{\sigma \to 0} \int_\sigma \left(\psi \frac{\partial \Omega}{\partial n} - \Omega \frac{\partial \psi}{\partial n}\right) \mathrm{d}\sigma \tag{3}$$

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